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NEW STABLE NUMERICAL INVERSION OF ABEL'S INTEGRAL EQUATION

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Abstract—The 3-D image reconstruction from cone-beam projections in computerized tomography leads naturally, in the case of radial symmetry, to the study of Abel's type of integral equations. A new method for the numerical solution of such equations when the experimental information is obtained through measured data, on a discrete set of points, is presented and rigorously analyzed.

1. INTRODUCTION

The difficult problem of determining the structure of an object from its 3-D cone-beam data projections is currently receiving considerable attention.

When the object is known to be radially symmetric, its structure can be determined by using the inverse Abel transform. If the object does not have radial symmetry, it can be reconstructed, in principle, by using the inverse Radon transform.

The purpose of this paper is to present and analyze a new stable method for the numerical solution of Abel's integral equation, based on a reconstruction technique initially proposed by B.K.P. Horn [1] for arbitrary 2-D ray-sampling schemes and more recently extended to 3-D image reconstruction methods from cone-beam projections by B.D. Smith [2].

Abel's integral equation can be written as

$$f(x) = \int_0^x g(s)\,(x-s)^{-\frac{1}{2}}\,ds, \qquad 0 \le x \le 1, \tag{1}$$

where the function $f(x)$ is the data function and $g(s)$ is the unknown function. The exact solution, see for instance Tricomi [3], is given by

$$g(x) = \frac{1}{\pi}\int_0^x f'(s)\,(x-s)^{-\frac{1}{2}}\,ds, \qquad 0 \le x \le 1, \tag{2}$$

assuming, without loss of generality, $f(0) = 0$.

It is well known that the process of estimating the solution function $g(x)$ if the data function $f(x)$ is given approximately and only at a discrete set of data points is ill-posed because small errors in the data $f(x)$ might cause large errors in the computed solution $g(x)$. Consequently, the direct use of formula (2) is very limited and special methods are needed.

The numerical solution of Abel's integral equation has been discussed by many authors and a number of different solutions methods have been proposed. Orthogonal functions approaches have been used, for example, [4–5]. Product integration techniques were utilized [6–7]. Regularization procedures have been analyzed by [8] and more recently by [9].

In Section 2, we describe the stabilized problem, analyze the consistency and stability properties of the new algorithm and obtain a sharp upper bound for the error. In Section 3, we study in detail the discrete numerical implementation of the method and, in Section 4, we show several computational results of interest. We also include a summary and some conclusions.

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2. STABILIZED PROBLEM

Let $C^0(I)$ denote the set of continuous functions over the interval $I = [0,1]$ with $\|f\|_{\infty,I} = \max_{x\in I}|f(x)| < \infty$.

We consider the problem of estimating in I, the inverse Abel transform $g(x)$ of a function $f(x)$ defined on I and observed with error. We assume that $f(x)$ is twice continuously differentiable on I. Instead of $f(x)$, we know some data function $f^\epsilon(x) \in C^0(I)$ such that $\|f^\epsilon - f\|_{\infty,I} \leq \epsilon$.

For errorless data, after integrating by parts in Equation (2), we obtain the equivalent solution expressions

$$g(x) = \frac{1}{\pi}\left\{2x^{\frac{1}{2}}f'(0) + 2\int_0^x f''(s)(x-s)^{\frac{1}{2}}\,ds\right\}, \qquad 0 \leq x \leq 1, \tag{3}$$

and

$$g(x) = \frac{1}{\pi}\lim_{\alpha\to 0}\left\{\alpha^{-\frac{1}{2}}f(x-\alpha) - \frac{1}{2}\int_0^{x-\alpha} f(s)(x-s)^{-\frac{3}{2}}ds\right\}, \qquad 0 \leq x \leq 1. \tag{4}$$

Following B.K.P. Horn [1], we observe that

$$\alpha^{-\frac{1}{2}}f(x-\alpha) - \alpha^{-\frac{3}{2}}\int_{x-\alpha}^x f(s)\,ds = -\frac{1}{2}\alpha^{\frac{1}{2}}f'(x) + 0(\alpha^{\frac{3}{2}}) \tag{5}$$

and

$$\frac{1}{\pi}\lim_{\alpha\to 0}\left\{\alpha^{-\frac{1}{2}}f(x-\alpha) - \alpha^{-\frac{3}{2}}\int_{x-\alpha}^x f(s)\,ds\right\} = 0. \tag{6}$$

Subtracting Equation (6) from Equation (4) gives

$$g(x) = \frac{1}{\pi}\lim_{\alpha\to 0}\left\{\alpha^{-\frac{3}{2}}\int_{x-\alpha}^x f(s)\,ds - \frac{1}{2}\int_0^{x-\alpha} f(s)(x-s)^{-\frac{3}{2}}\,ds\right\}, \qquad 0 \leq x \leq 1.$$

The approximate inverse Abel transform is now obtained by eliminating the limit procedure in the last expression, i.e.,

$$g_\alpha(x) = \frac{1}{\pi}\left\{\alpha^{-\frac{3}{2}}\int_{x-\alpha}^x f(s)\,ds - \frac{1}{2}\int_0^{x-\alpha} f(s)(x-s)^{-\frac{3}{2}}\,ds\right\}, \qquad 0 \leq x \leq 1. \tag{7}$$

Notice that by defining the kernel function $(0 < \alpha < 1)$

$$H_\alpha(t) = \begin{cases} \alpha^{-\frac{3}{2}}, & 0 \leq t < \alpha, \\ -\frac{1}{2}t^{-\frac{3}{2}}, & \alpha \leq t, \end{cases} \tag{8}$$

the approximate solution can also be written as

$$g_\alpha(x) = \frac{1}{\pi}(H_\alpha * f)(x), \qquad 0 \leq x \leq 1. \tag{9}$$

The family of kernel functions $H_\alpha(t)$ satisfy the normalizing property $\int_0^1 H_\alpha(t)dt = 1$, $0 < \alpha < 1$.

The following two lemmas are fundamental for our results.

LEMMA 1. *(Consistency). If $f(0) = 0$, $\|f'\|_{\infty,I} \leq M_1$ and $f'' \in C^0(I)$, then $\|g - g_\alpha\|_{\infty,I} \leq \frac{5}{2}\frac{\alpha^{\frac{1}{2}}}{\pi}M_1 + 0(\alpha^{\frac{3}{2}})$.*

PROOF. Integrating by parts in Equation (7) and using Equation (5) plus the fact that $f(0) = 0$, we obtain

$$g_\alpha(x) = \frac{1}{\pi}\left\{\frac{1}{2}\alpha^{\frac{1}{2}}f'(x) + 0(\alpha^{\frac{3}{2}}) + \int_0^{x-\alpha} f'(s)(x-s)^{-\frac{1}{2}}\,ds\right\}, \qquad 0 \leq x \leq 1.$$

Integrating by parts again, recalling that $f'' \in C^0(I)$,

$$g_\alpha(x) = \frac{1}{\pi}\left\{\frac{1}{2}\alpha^{\frac{1}{2}}f'(x) - \frac{2}{\pi}\alpha^{\frac{1}{2}}f'(x-\alpha) + \frac{2}{\pi}f'(0)x^{\frac{1}{2}} + 2\int_0^{x-\alpha} f''(s)(x-s)^{\frac{1}{2}}ds + 0(\alpha^{\frac{3}{2}})\right\}, \qquad 0 \leq x \leq 1.$$

A comparison of this formula with expression (3) allows us to write

$$g_\alpha(x) - g(x) = \frac{1}{\pi}\left\{\alpha^{\frac{1}{2}}\left(\frac{1}{2}f'(x) - 2f'(x-\alpha)\right) - 2\int_{x-\alpha}^{x} f''(s)(x-s)^{\frac{1}{2}}\,ds\right\} + 0(\alpha^{\frac{3}{2}}), \qquad 0 \leq x \leq 1.$$

Observing that $|\frac{2}{\pi}\int_{x-\alpha}^{x} f''(s)(x-s)^{\frac{1}{2}}ds| \leq \frac{4}{3}\frac{\alpha^{\frac{3}{2}}}{\pi}\|f''\|_{\infty,I}$ and $|\frac{1}{2}f'(x) - 2f'(x-\alpha)| \leq \frac{1}{2}|f'(x)| + 2|f'(x)| + 2\alpha\|f''\|_{\infty,I}$, it follows that

$$|g_\alpha(x) - g(x)| \leq \frac{5}{2}\frac{\alpha^{\frac{1}{2}}}{\pi}|f'(x)| + 0(\alpha^{\frac{3}{2}}), \qquad 0 \leq x \leq 1.$$

Thus,

$$\|g_\alpha - g\|_{\infty,I} \leq \frac{5}{2}\frac{\alpha^{\frac{1}{2}}}{\pi}M_1 + 0(\alpha^{\frac{3}{2}}).$$

LEMMA 2. *(Stability). If $f^\epsilon \in C^0(I)$ and $\|f - f^\epsilon\|_{\infty,I} \leq \epsilon$, then $\|g_\alpha - g_\alpha^\epsilon\|_{\infty,I} \leq \frac{2}{\pi}\epsilon\alpha^{-\frac{1}{2}}$.*
PROOF. Using (7), we get

$$g_\alpha^\epsilon(x) - g_\alpha(x) = \frac{1}{\pi}\left\{\alpha^{-\frac{3}{2}}\int_{x-\alpha}^{x}(f^\epsilon(s) - f(s))\,ds - \frac{1}{2}\int_0^{x-\alpha}(f^\epsilon(s) - f(s))(x-s)^{-\frac{3}{2}}ds\right\}, \qquad 0 \leq x \leq 1.$$

Hence,

$$|g_\alpha^\epsilon(x) - g_\alpha(x)| \leq \frac{\epsilon}{\pi}\left\{\alpha^{-\frac{1}{2}} + (\alpha^{-\frac{1}{2}} - x^{-\frac{1}{2}})\right\} = \frac{2}{\pi}\epsilon\alpha^{-\frac{1}{2}}\left(1 - \frac{1}{2}\alpha^{\frac{1}{2}}x^{-\frac{1}{2}}\right) \leq \frac{2}{\pi}\epsilon\alpha^{-\frac{1}{2}}, \qquad 0 \leq x \leq 1,$$

and

$$\|g_\alpha^\epsilon - g_\alpha\|_{\infty,I} \leq \frac{2}{\pi}\epsilon\alpha^{-\frac{1}{2}}.$$

Lemma 2 shows that attempting to reconstruct the approximate inverse Abel transform function g_α is a stable problem with respect to perturbations in the data function, in the maximum norm, for α fixed.

THEOREM 1. *(Error Estimate) Under the conditions of Lemmas 1 and 2,*

$$\|g_\alpha^\epsilon - g\|_{\infty,I} \leq \frac{5}{2}\frac{\alpha^{\frac{1}{2}}}{\pi}M_1 + \frac{2}{\pi}\epsilon\alpha^{-\frac{1}{2}} + 0(\alpha^{\frac{3}{2}}). \tag{10}$$

PROOF. The estimate follows from Lemmas 1 and 2 and the triangle inequality.

We observe that the principal term of the error estimate (10) is minimized by choosing $\alpha = \bar{\alpha} = 4/5\epsilon/M_1$. For this "optimal" choice of the regularization parameter, the error estimate becomes

$$\|g_{\bar{\alpha}}^\epsilon - g\|_{\infty,I} \leq \frac{2\sqrt{5}}{\pi}M_1^{\frac{1}{2}}\epsilon^{\frac{1}{2}} + 0(\epsilon^{\frac{3}{2}})$$

and we obtain uniform convergence as $\epsilon \longrightarrow 0$ with rate $0(\epsilon^{1/2})$.

3. NUMERICAL PROCEDURE

To numerically approximate $g^\epsilon_\alpha(x)$, a quadrature formula for the convolution Equation (9) is needed. In order to perform the discrete error analysis we shall further assume the data function $f^\epsilon(x)$ to be uniformly Lipschitz on $I = [0,1]$, with Lipschitz constant L.

For $h > 0$ and $Nh = 1$, we let $s_i = ih$ and denote $f^\epsilon(s_i) = f^\epsilon_i$, $i = 0,1,\dots,N$, with $f^\epsilon_0 = 0$. The objective is to introduce a simple quadrature formula to avoid any artificial smoothing in the process. To that effect, given x_j, $j = 0,1,\dots,N$, we define $p^\epsilon(x) = \sum_{i=0}^{j} f^\epsilon_i \phi_i(x)$, $0 \le x \le x_j$, a piecewise interpolant of $f^\epsilon(x)$ at the grid points x_j. Here,

$$\varphi_0(x) = \begin{cases} 1, & 0 \le x \le \frac{h}{2} \\ 0 & \text{otherwise}, \end{cases} \qquad \varphi_j(x) = \begin{cases} 1, & x_j - \frac{h}{2} \le x \le x_j \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\varphi_i(x) = \begin{cases} 1, & x_i - \frac{h}{2} \le x \le x_i + \frac{h}{2} \\ 0 & \text{otherwise}, \end{cases}$$

or

$$\varphi_0(x) = \begin{cases} 1 - \frac{x}{h}, & 0 \le x \le h \\ 0 & \text{otherwise}, \end{cases} \qquad \varphi_j(x) = \begin{cases} 1 + \frac{(x-x_j)}{h}, & x_{j-1} \le x \le x_j \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi_i(x) = \begin{cases} 1 + \frac{(x-x_i)}{h}, & x_{i-1} \le x \le x_i \\ 1 - \frac{(x-x_i)}{h}, & x_i \le x \le x_{i+1}, \quad 0 < i < j \le N, \end{cases}$$

corresponding to piecewise constant interpolation or piecewise linear interpolation respectively.

The quadrature formula is obtained by directly convolving the kernel function H_α against p^ϵ as follows:

$$G^\epsilon_{\alpha,h}(x_j) = \frac{1}{\pi}(H_\alpha * p^\epsilon)(x_j) = \frac{1}{\pi}\int_0^{x_j} H_\alpha(x_j - s)p^\epsilon(s)\,ds = \frac{1}{\pi}\sum_{i=0}^{j} f^\epsilon_i b^\alpha_i(x_j), \tag{11}$$

where

$$b^\alpha_i(x_j) = \int_0^{x_j} H_\alpha(x_j - s)\varphi_i(s)\,ds$$

is evaluated exactly for $i = 0,1,\dots,j$.

Thus, we have

$$\begin{aligned} G^\epsilon_{\alpha,h}(0) &= 0 \\ G^\epsilon_{\alpha,h}(x_j) &= \frac{1}{\pi}\sum_{i=0}^{j} f^\epsilon_i b^\alpha_i(x_j), \qquad j = 1,2,\dots,N. \end{aligned} \tag{12}$$

From Equations (9) and (11) it follows that

$$\begin{aligned} |g^\epsilon_\alpha(x_j) - G^\epsilon_{\alpha,h}(x_j)| &= \frac{1}{\pi}\left| \int_0^{x_j} H_\alpha(x_j - s)(f^\epsilon(s) - p^\epsilon(s))\,ds \right| \\ &\le \frac{1}{\pi}\max_{0\le s\le x_j} |f^\epsilon(s) - p^\epsilon(s)| \int_0^{x_j} |H_\alpha(x_j - s)|\,ds. \end{aligned} \tag{13}$$

From the definition (8) of the kernel function H_α, we immediately obtain

$$\int_0^{x_j} |H_\alpha(x_j - s)|\,ds = 2\alpha^{-\frac{1}{2}} - x_j^{-\frac{1}{2}} \le 2\,\alpha^{-\frac{1}{2}}. \tag{14}$$

Recalling the Lipschitz property of f^ϵ and the fact that p^ϵ is a piecewise interpolant of f^ϵ at the grid points, we have

$$\max_{0\le s\le x_j} |f^\epsilon(s) - p^\epsilon(s)| = \max_{i=1,2,\dots,j}\left\{ \max_{x_{i-1}\le s\le x_i} |f^\epsilon(s) - p^\epsilon(s)| \right\} \le Lh. \tag{15}$$

Using estimates (14) and (15) in inequality (13), we get

$$|g_\alpha^\epsilon(x_j) - G_{\alpha,h}^\epsilon(x_j)| \le \frac{2}{\pi} L\alpha^{-\frac{1}{2}} h, \qquad j = 0, 1, \ldots, N.$$

Combining this upper bound with the restriction to the grid points of the error estimate (10) and using the triangle inequality, we have the following theorem.

THEOREM 2. *(Discrete error estimate) Under the conditions of Theorem 1, if $f^\epsilon(x)$ is uniformly Lipschitz on $I = [0,1]$ with Lipschitz constant L and if the approximate solution $G_{\alpha,h}^\epsilon(x_j)$ is computed using formula (12), then*

$$\max_{0\le j\le N} |G_{\alpha,h}^\epsilon(x_j) - g(x_j)| \le \frac{5}{2} \frac{\alpha^{\frac{1}{2}}}{\pi} M_1 + 2\frac{\alpha^{-12}}{\pi} (\epsilon + Lh) + 0(\alpha^{\frac{3}{2}}).$$

The error estimate for the discrete case is obtained by adding the global truncation error to the error estimate of the non-discrete case.

4. NUMERICAL RESULTS

In this section, we discuss the implementation of our numerical method and the tests which we have performed in order to investigate the accuracy and stability of the numerical inverse Abel's integral equation procedure.

In all the examples, the exact data function is denoted $f(x)$ and the noisy data function $f^\epsilon(x)$ is obtained by adding an ϵ random error to $f(x)$, i.e., $f^\epsilon(s_i) = f(s_i) + \epsilon\theta_i$, where $s_i = ih$, $i = 0, 1, \ldots, N$, $Nh = 1$ and θ_i is a uniform random variable with values in $[-1, 1]$ such that $\max_{0\le i\le N} |f^\epsilon(s_i) - f(s_i)| \le \epsilon$.

The discrete numerical approximation to the inverse Abel transform $g(x)$, denoted $G_{\alpha,h}^\epsilon(x_j)$, is reconstructed using Equation (12). In what follows, we use the discrete l_2 norm in $I = [0,1]$ to measure errors. This norm is defined by

$$\|f\|_{2,I} = \left[\frac{1}{N+1} \sum_{i=0}^{N} |f(x_i)|^2\right]^{\frac{1}{2}}$$

and we observe that the estimates of previous sections, obtained with respect to the discrete maximum norm, are applicable when using the discrete l_2 norm.

EXAMPLE 1. As a first example, we consider the data function $f(x) = x$ with exact inverse Abel transform $g(x) = 2/\pi x^{1/2}$. The data function $f(x)$ satisfies all the hypotheses of Theorem 1 in Section 2. Figure 1 shows the approximate solution obtained with our method using piecewise constant interpolation (dashed line) and the exact solution (full line) for $\epsilon = 0.001$, $N = 1000$ and $\alpha = 0.002$. The corresponding l_2 error norm is $\|G_{\alpha,h}^\epsilon - g\|_{2,I} = 0.0160$.

EXAMPLE 2. Our second example is for

$$f(x) = \begin{cases} 2x^2, & 0 \le x \le \frac{1}{2} \\ 1 - 2(1-x)^2, & \frac{1}{2} < x \le 1. \end{cases}$$

This data function is only once continuously differentiable on $I = [0,1]$. The second derivative is discontinuous at $x = 1/2$, partially violating the conditions of Theorem 1. We use $\epsilon = 0.005$, $N = 1000$ and $\alpha = 0.002$. In Figure 2, we plot the numerical solution obtained with our method using piecewise constant interpolation (dashed line) and the exact solution (full line), given by

$$g(x) = \begin{cases} \frac{16}{3}\frac{1}{\pi}x^{\frac{3}{2}}, & 0 \le x \le \frac{1}{2} \\ \frac{16}{3}\frac{1}{\pi}x^{\frac{3}{2}} + \frac{16}{3}\frac{1}{\pi}(x-\frac{1}{2})^{\frac{3}{2}} - \frac{8}{\pi}(x-\frac{1}{2})^{\frac{1}{2}}(2x-1), & \frac{1}{2} < x \le 1 \end{cases}$$

In this case, $\|G_{\alpha,h}^\epsilon - g\|_{2,I} = 0.0253$.

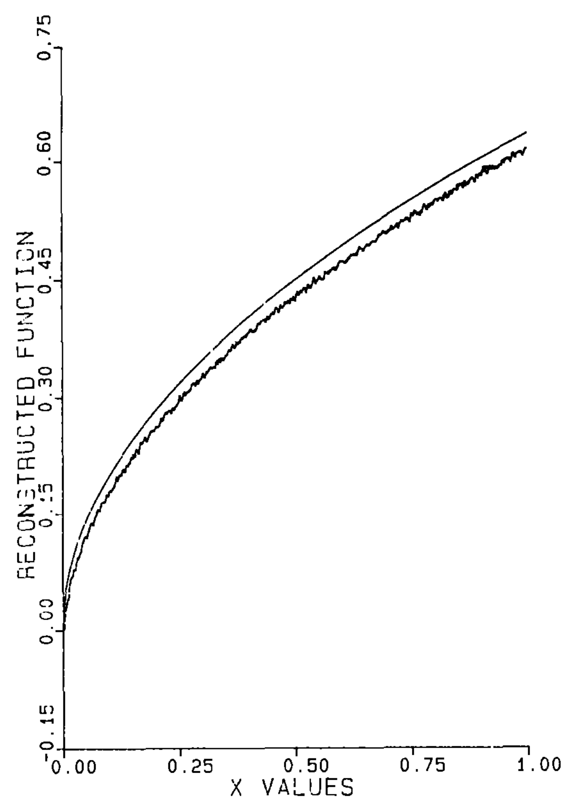

Figure 1. Reconstructed inverse Abel transform, Example 1 (Piecewise constant approximation). ϵ = 0.001, α = 0.002, N = 1000.

EXAMPLE 3. In Figure 3, we show the exact inverse Abel transform function $g(x)$ (full line) and the approximate solution obtained with our method using piecewise constant interpolation (dashed line). In this example, $\epsilon = 0.01$, $N = 1000$, $\alpha = 0.002$ and $\|G^{\epsilon}_{\alpha,h} - g\|_{2,I} = 0.0792$. The exact data function is given by

$$f(x) = \begin{cases} 0, & 0 \leq x < 0.2 \\ 2(x - 0.2)^{\frac{1}{2}}, & 0.2 \leq x < 0.6 \\ 2(x - 0.2)^{\frac{1}{2}} - 2(x - 0.6)^{\frac{1}{2}}, & 0.6 \leq x \leq 1 \end{cases}$$

and the exact solution is the function

$$g(x) = \begin{cases} 1, & 0.2 \leq x \leq 0.6 \\ 0, & \text{otherwise.} \end{cases}$$

The first derivative of $f(x)$ is not continuous on $I = [0, 1]$, strongly violating all the hypotheses of Theorem 1. However, this reconstruction constitutes an important challenging test for the practical utilization of the method.

The numerical stabililty property of the algorithm is illustrated in Tables 1(1′), 2(2′), 3(3′) where the discrete l_2 norm of the error is shown as a function of the amount of noise in the data ϵ for Examples 1, 2, and 3, respectively, using piecewise constant (linear) interpolation.

We notice that in all cases, the numerical stability of the method is confirmed. Moreover, in the ϵ-range 0.000 to 0.010, the error norms are barely sensitive to changes in $h = 1/N$ in Examples 1 and 2 and only slightly more sensitive to changes in h in Example 3.

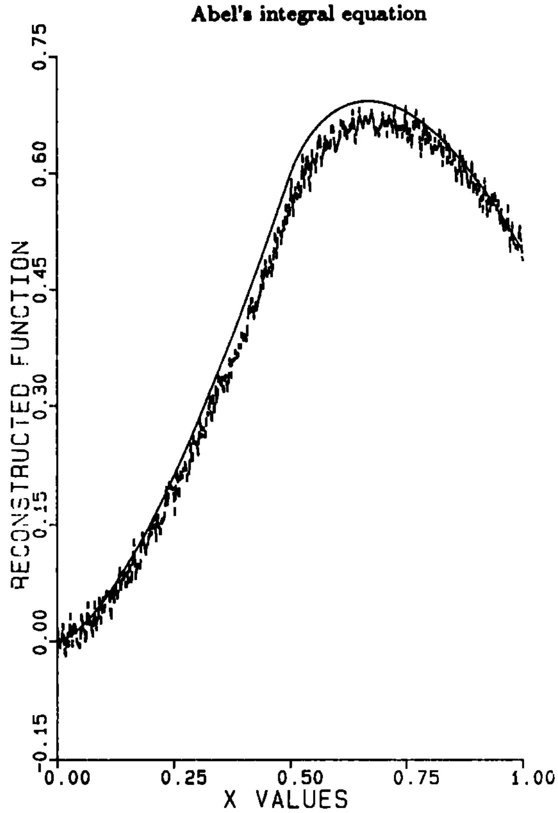

Figure 2. Reconstructed inverse Abel transform, Example 2 (Piecewise constant approximation). $\epsilon = 0.005$, $\alpha = 0.002$, $N = 1000$.

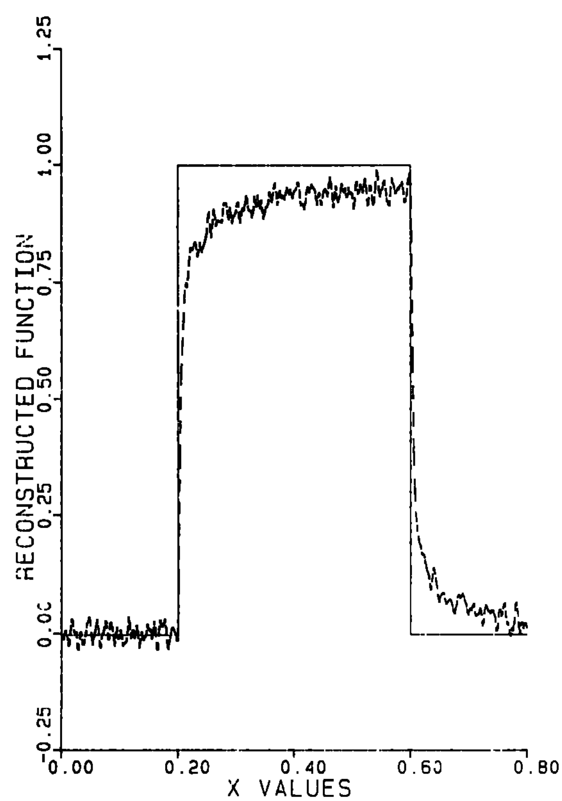

Figure 3. Reconstructed inverse Abel transform, Example 3 (Piecewise constant approximation). $\epsilon = 0.010$, $\alpha = 0.002$, $N = 1000$.

Table 1. Error norms as functions of ϵ in Example 1, using piecewise constant interpolation.

Example 1	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
ϵ	$\alpha = 0.0025$	$\alpha = 0.004$	$\alpha = 0.001$
0.000	0.0136	0.0216	0.0153
0.001	0.0142	0.0218	0.0157
0.005	0.0241	0.0252	0.0237
0.010	0.0422	0.0250	0.0394

Table 1′. Error norms as functions of ϵ in Example 1, using piecewise linear interpolation.

Example 1	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
ϵ	$\alpha = 0.0025$	$\alpha = 0.004$	$\alpha = 0.001$
0.000	0.0239	0.0302	0.0151
0.001	0.0240	0.0303	0.0155
0.005	0.0269	0.0315	0.0238
0.010	0.0346	0.0349	0.0398

Table 2. Error norms as functions of ϵ in Example 2, using piecewise constant interpolation.

Example 2	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
ϵ	$\alpha = 0.0025$	$\alpha = 0.001$	$\alpha = 0.002$
0.000	0.0157	0.0099	0.0177
0.001	0.0162	0.0117	0.0180
0.005	0.0254	0.0327	0.0253
0.010	0.0430	0.0630	0.0349

Table 2′. Error norms as functions of ϵ in Example 2, using piecewise linear interpolation.

Example 2	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
ϵ	$\alpha = 0.0025$	$\alpha = 0.001$	$\alpha = 0.002$
0.000	0.0276	0.0174	0.0247
0.001	0.0277	0.0179	0.0248
0.005	0.0304	0.0263	0.0277
0.010	0.0376	0.0431	0.0351

Table 3. Error norms as functions of ϵ in Example 3, using piecewise linear interpolation.

Example 3	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
ϵ	$\alpha = 0.0025$	$\alpha = 0.001$	$\alpha = 0.0005$
0.000	0.0311	0.0501	0.0173
0.001	0.0313	0.0504	0.0194
0.005	0.0368	0.0588	0.0469
0.010	0.0504	0.0796	0.0887

Table 3′. Error norms as functions of ϵ in Example 3, using piecewise linear interpolation.

Example 3	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
ϵ	$\alpha = 0.0025$	$\alpha = 0.001$	$\alpha = 0.0005$
0.000	0.0672	0.0648	0.0354
0.001	0.0672	0.0649	0.0357
0.005	0.0677	0.0678	0.0445
0.010	0.0705	0.0760	0.0651

Table 4. Error norms as functions of α in Example 1, using piecewise linear interpolation.

Example 1	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
α	$\epsilon = 0.01$	$\epsilon = 0.01$	$\epsilon = 0.01$
0.001	0.0422	0.0629	0.0872
0.002	0.0421	0.0337	0.0394
0.004	0.0352	0.0360	0.0326
0.006	0.0400	0.0387	0.0330
0.008	0.0432	0.0421	0.0403

Table 4′. Error norms as functions of α in Example 1, using piecewise linear interpolation.

Example 1	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
α	$\epsilon = 0.01$	$\epsilon = 0.01$	$\epsilon = 0.01$
0.001	0.0400	0.0421	0.0398
0.002	0.0354	0.0338	0.0326
0.004	0.0353	0.0349	0.0330
0.006	0.0400	0.0391	0.0381
0.008	0.0445	0.0439	0.0432

Table 5. Error norms as functions of α in Example 1, using piecewise linear interpolation.

Example 1	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
α	$\epsilon = 0.01$	$\epsilon = 0.01$	$\epsilon = 0.01$
0.001	0.0530	0.0630	0.0873
0.002	0.0429	0.0358	0.0404
0.004	0.0393	0.0389	0.0349
0.006	0.0454	0.0444	0.0365
0.008	0.0490	0.0502	0.0398

Table 5′. Error norms as functions of α in Example 2, using piecewise linear interpolation.

Example 2	ERROR NORMS $N = 200$	$N = 500$	$N = 1000$
α	$\epsilon = 0.01$	$\epsilon = 0.01$	$\epsilon = 0.01$
0.001	0.0412	0.0431	0.0409
0.002	0.0379	0.0360	0.0351
0.004	0.0398	0.0391	0.0376
0.006	0.0458	0.0446	0.0439
0.008	0.0513	0.0503	0.0500

Table 6. Error norms as functions of α in Example 3, using piecewise linear interpolation.

Example	ERROR NORMS		
3	$N = 200$	$N = 500$	$N = 1000$
α	$\epsilon = 0.01$	$\epsilon = 0.01$	$\epsilon = 0.01$
0.001	0.0504	0.0796	0.0640
0.002	0.0504	0.0892	0.0792
0.004	0.0889	0.1140	0.1059
0.006	0.1145	0.1333	0.1256
0.008	0.1247	0.1493	0.1542

Table 6′. Error norms as functions of α in Example 3, using piecewise linear interpolation.

Example	ERROR NORMS		
3	$N = 200$	$N = 500$	$N = 1000$
α	$\epsilon = 0.01$	$\epsilon = 0.01$	$\epsilon = 0.01$
0.001	0.0513	0.0760	0.0638
0.002	0.0635	0.0887	0.0790
0.004	0.0910	0.1138	0.1058
0.006	0.1148	0.1332	0.1264
0.008	0.1326	0.1492	0.1434

Tables 4(4′), 5(5′), 6(6′) illustrate the behavior of the discrete l_2 error norm as a function of the regularization parameter α for Examples 1, 2 and 3, respectively. We only show the results for $\epsilon = 0.01$ which are typical.

In all cases, a choice of α in the interval $[0.002, 0.004]$ gives quite acceptable accuracy in the presence of high level of noise. For smaller values of ϵ ($\epsilon < 0.01$), a choice of α between 0.001 and 0.003 produces good resolution while ensuring numerical stability.

5. SUMMARY AND CONCLUSIONS

A new method for the numerical solution of Abel's type of integral equations is derived and analyzed. The stability with respect to the data is restored and good accuracy is obtained, even for small sample intervals and high noise levels in the data.

Several test cases are investigated using piecewise constant and piecewise linear interpolations. A number of parameters are varied, including number of sampling points N, amount of noise in the data ϵ, and regularization parameter α. In all cases, the computed solution functions are independent of N (over the range tested) for fixed ϵ and α.

Piecewise constant approximations of the reconstructed function are compared with the corresponding piecewise linear approximations and found to be quite similar. However, for small values of $\alpha(0 < \alpha \leq h/2)$, the former reconstructions are independent of α.

A general conclusion is that the choice of the regularization parameter is not critical. Nevertheless, it is always possible to automatically select a particular value of α—that allows for the computation of a solution whose quality is in agreement with the quality of the input data—by approximately minimizing the discrepancy quantity $|\|G^{\epsilon}_{\alpha,h} * x^{-1/2}\|_{2,I} - 3\epsilon|$ over a suitable monotone sequence of α values.

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